CCRT: Categorical and Combinatorial Representation Theory.

From combinatorics of universal problems
to usual applications.

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Collaboration at various stages of the work
and in the framework of the Project
Evolution Equations in Combinatorics and Physics :
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CIP seminar,

Friday conversations:

For this seminar, please have a look at Slide CCRT[n] & ff.

Goal of this series of talks

The goal of these talks is threefold

1. Category theory aimed at "free formulas" and their combinatorics
2. How to construct free objects
    1. w.r.t. a functor with - at least - two combinatorial applications:
        1. the two routes to reach the free algebra
        2. alphabets interpolating between commutative and non commutative worlds
    2. without functor: sums, tensor and free products
    3. w.r.t. a diagram: limits
3. Representation theory: Categories of modules, semi-simplicity, isomorphism classes i.e. the framework of Kronecker coefficients.
4. MRS factorisation: A local system of coordinates for Hausdorff groups.

CCRT[11] Partially Commutative structures from a functorial point of view and MRS factorisations.

Preamble. – Today, we will consider four categories:

$$\mathbf{Mon},\ \mathbf{Grp},\ \mathbf{k\text{-}Lie},\ \mathbf{k\text{-}AAU} \tag{1}$$

In each of these categories, there is a notion of "What are two commuting elements"

- in **Mon**, **Grp**, **k-AAU**, it is $xy = yx$
- in **k-Lie** it is $[x, y] = 0$

but, for all of them, this relation is *reflexive* and *symmetric*.
This leads us to the following questions

1. What is the best system or category of formal generators ?

Plan

1. Monoid (pc words, counting, Cartier-Foata, fine grading)
2. Group (reduced words)
3. **k-AAU** (double structure: as enveloping algebra and as monoid algebra)
4. **k-Lie**
5. MRS Factorisation (arbitrary basis and order)
6. Möbius functions in general
7. Möbius functions for Free PC monoid
8. Closure theorem
9. Some concluding remarks

2. By “category of formal generators”, we mean, in the noncommutative world we have noncommutative alphabets and words, in the fully commutative world, have indeterminates (commutative alphabets) and monomials (with multiindex power notation).
3. What is the combinatorics of these formulas ?
4. What is Lazard elimination ?
5. In what generality does MRS factorization hold ?
6. What is Magnus theory ? What are its arenas ?
7. What are the characters here ? and Möbius functions ?

First remarks/1

1. As a motivation, we will begin by answering question 7 (the last one), and by very simple examples.
2. Let us first consider the $\mathbf{k}$-algebra $\mathbf{k}\langle x, y\rangle = \mathbf{k}[\{x, y\}^*]$ of non-commutative polynomials in the two noncommuting variables x, y over $\mathbf{k}$.
   The character of $\mathbf{k}\langle x, y\rangle$ that sends x and y to α and β is explicitly given as the Kleene star
   $$(\alpha.x + \beta.y)^* = \sum_{n\geq 0} (\alpha.x + \beta.y)^n.$$
3. Consider now the $\mathbf{k}$-algebra $\mathbf{k}[x, y] = \mathbf{k}[\{x^p y^q\}_{p,q\in\mathbb{N}}]$ of commutative polynomials in two (commuting) variables x, y over $\mathbf{k}$. As $\mathbf{k}$ is commutative, a character of this $\mathbf{k}$-algebra is uniquely determined by the images α and β of x and y. Such a character is again determined by a Kleene star. Indeed

First remarks/2

4. We remark that these two algebras share a common feature: *they are algebras of monoids*, so we will consider this question in general and see that it covers the celebrated Möbius arithmetic function.
5. We remark also that commutations can be formulated as relations between words. So we embarks towards the notion of *monoidal relation*.

The category **Mon** through the looking glass/1.

1. We recall that **Mon**, the category of monoids, is defined by monoids as objects and unit-and-products preserving maps between them as arrows.
2. When one want to adress the question "What is a kernel in the category **Mon**", we see that the usual definition through pullbacks [33] does not work in it. So we have to look closer to the equivalences indiced by morphisms.
3. One can prove the following

Theorem (Th 1)

Let $f: M \to N$ be a morphism of monoids and $\equiv_f$ be the equivalence relation

$$x_1 \equiv_f x_2 \overset{def}{\Longleftrightarrow} f(x_1) = f(x_2) \tag{2}$$

i) Then

$$(\forall s, t \in M)(x_1 \equiv_f x_2 \Longrightarrow s.x_1.t \equiv_f s.x_2.t) \qquad (Cong) \tag{3}$$

The category **Mon** through the looking glass/2.

Theorem (Th 1 cont'd)

ii) Conversely, given a monoid M and an equivalence relation $\equiv$ on $M \times M$ satisfying (Cong) above there exists a unique structure of monoid on $M/\equiv$ such that s, the (set-theoretical) canonical map $M \to M/\equiv$ be a morphism of monoids (of course, in this case, $\equiv_s$ equals $\equiv$).

4. We will call <u>congruences</u> equivalence relations on a monoid satisfying the property (Cong) of eq. (3). We have

Theorem (Th 1 cont'd/3)

iii) The sublattice $EqCong(M)$ of $EqRel(M)$[a] is closed by arbitrary (i.e. finite or infinite) intersections.

[a] $EqRel(X)$ is the set of all equivalence relations on a set X. It is a subset of $X \times X$ closed by (finite or infinite) intersections.

Presentations

5. The preceding study help us to define a monoid presented by generators and relations.
6. A (monoidal) relator is a set of pairs words $\mathbf{R} = \{(u_i, v_i)\}_{i \in I}$
7. The congruence generated by $\mathbf{R}$, is the congruence $\equiv_{\mathbf{R}}$ is the intersection of all congruences such that

$$(u, v) \in \mathbf{R} \Longrightarrow u \equiv v \tag{4}$$

we then define

$$\equiv_{\mathbf{R}} := \bigcap_{\substack{\equiv \in EqCong(M) \\ u \equiv v \text{ for } (u,v) \in \mathbf{R}}} \equiv \tag{5}$$

8. and

$$\langle X; \mathbf{R} \rangle_{\mathbf{Mon}} := X^* / \equiv_{\mathbf{R}} \tag{6}$$

Counting the words

9. Take a total ordering on the alphabet $X = \{x_1, \ldots, x_n\}$ increasingly and X^* by the graded lexicographic order $\prec_{grlex}$ (left to right) defined by

$$u \prec_{grlex} v \iff |u| < |v| \text{ or } u = p{\color{red}x}s_1,\;\; u = p{\color{red}y}s_2 \text{ with } x < y \tag{7}$$

10. Order $\mathbf{R}$ such that $u \prec_{grlex} v$ for all $(u, v) \in \mathbf{R}$.
11. Construct the following sequence

$$\begin{array}{rcl}
P_0 := \{1_{X^*}\} & ; & W_{(0,0)} = \{1_{X^*}\} = X^0; \\
\vdots \quad \vdots \quad \vdots & ; & \vdots \quad \vdots \quad \vdots \\
P_n & ; & W_{(n,0)} = W_{n,max(n-1)} \cdots W_{n,max(n)}, \\
 & ; & W_n = \cup_{0 \le j \le max(n)} W_{(n,j)} \\
\text{consider all } {\color{red}x} W_n,\;\; x \in X \text{ and} & ; & \\
\text{eliminate all } v \text{ with } (u, v) \in \mathbf{R} & ; & P_{n+1} = P_n \cup W_n
\end{array}$$

Counting the words/2

Example of the symmetric group

12. The symmetric group $\mathfrak{S}_n$ can be defined by the Moore-Coxeter presentation

$$\langle \{t_1, t_2, \cdots, t_{n-1}\};\ t_i^2 = 1,\ t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1} \rangle_{\mathbf{Mon}} \tag{8}$$

13. For example $\mathfrak{S}_3 = \langle \{t_1, t_2\};\ t_i^2 = 1,\ t_1 t_2 t_1 = t_2 t_1 t_2 \rangle_{\mathbf{Mon}}$

14. The algorithm gives

$$\begin{array}{rcl}
P_0 := \{1_{X^*}\} & ; & W_{(0,0)} = \{1_{X^*}\} = X^0; \\
 & ; & W_{(1,1)} = \{t_1\}\ W_{(1,2)} = \{t_2\} \\
 & ; & W_1 = \{t_1, t_2\} \\
P_1 := \{1_{X^*}, t_1, t_2\} & ; & W_{2,1} = \{\color{red}{t_1 t_1}, t_1 t_2\}, W_{2,2} = \{t_2 t_1, \color{red}{t_2 t_2}\} \\
 & ; & W_2 = \{t_1 t_2, t_2 t_1\} \\
P_2 := \{1_{X^*}, t_1, t_2, t_1 t_2, t_2 t_1\} & ; & W_{3,1} = \{\color{red}{t_1 t_1 t_2}, t_1 t_2 t_1\}, \\
W_{3,2} & = & \{\color{red}{t_2 t_1 t_2}, \color{red}{t_2 t_2 t_1}\}, W_3 = \{t_1 t_2 t_1\} \\
P_3 := \{1_{X^*}, t_1, t_2, t_1 t_2, t_2 t_1, t_1 t_2 t_1\} & ; & \text{and then stop because } W_4 = \emptyset
\end{array}$$

Counting the words/3

15. Let us further consider the (square-free) monoid

$$\langle \{a, b\};\ a^2 = b^2 = 1\rangle_{\mathbf{Mon}} \tag{9}$$

16. The algorithm gives

$$\begin{aligned}
P_0 := \{1_{X^*}\} \quad &; \quad W_{(0,0)} = \{1_{X^*}\} = X^0;\\
&; \quad W_{(1,1)} = \{a\}\ W_{(1,2)} = \{b\}\\
&; \quad W_1 = \{a, b\}\\
P_1 := \{1_{X^*}, a, b\} \quad &; \quad W_{2,1} = \{{\color{red}aa}, ab\}, W_{2,2} = \{ba, {\color{red}bb}\}\\
&; \quad W_2 = \{ab, ba\}\\
P_2 := \{1_{X^*}, a, b, ab, ba\} \quad &; \quad W_{3,1} = \{{\color{red}aab}, aba\},\\
W_{3,2} \quad &= \quad \{bab, {\color{red}bba}\}, W_3 = \{aba, bab\}\\
&; \quad \text{never stops, normal forms } a(ba)^*, b(ab)^*
\end{aligned}$$

17. Enumeration $M_0 = 1;\ M_{n+1} = \{a(ba)^n, b(ab)^n\}$. Hilbert series $T = \sum_{n\geq 0} |M_n|.t^n$ is here $T = 1 + \dfrac{2x}{1-x} = \dfrac{1+x}{1-x}$

Counting the words: Hilbert Series

18. When the monoid M is finitely graded (i.e. $M = \uplus_{n\in\mathbb{N}} M_n,\ M_p.M_q \subset M_{p+q}$ and $|M_n| < +\infty$), we have a Hilbert series

$$Hilb(M,t) := \sum_{n\geq 0} |M_n|.t^n \tag{10}$$

for example, for the commutative monoid $M = \{x^{n_1}y^{n_2}z^{n_3}t^{n_4}\}_{n_i\in\mathbb{N}}$ (the one of monomials for the polynomials over the commutative alphabet $X = \{x,y,z,t\}$, graded by the length $|x^{n_1}y^{n_2}z^{n_3}t^{n_4}| = n_1+n_2+n_3+n_4$, the Hilbert series is

$$Hilb(M,l) = \frac{1}{1-4l+6l^2-4l^3+l^4} = \frac{1}{(1-l)^4} \tag{11}$$

Partially Commutative monoids

19. A partially commutative alphabet (X, θ) is a set endowed with a commutation relation $\theta \subset X \times X$, reflexive and symmetric.
20. The partially commutative monoid $M(X, \theta)$ is

$$M(X,\theta) := \langle X;\ (xy, yx)_{(x,y)\in\theta} \rangle_{\mathbf{Mon}} \tag{12}$$

21. If the alphabet is finite, we have

$$Hilb(M(X,\theta), t) = \frac{1}{\sum_{n\geq 0}(-1)^n c_n t^n} \tag{13}$$

where c_n is the number of n-cliques of θ. This is a consequence of a more general theorem of Cartier and Foata [5].

Where the (forgetful) functor comes: Monoids.

22. Def **CAlph** be the category of alphabets with commutation i.e. reflexive and symmetric graphs (X,θ) with $f:\ (X_1,\theta_1)\to(X_2,\theta_2)$ such that $f:\ X_1\to X_2$, set-theoretical such that $(u,v)\in\theta_1 \Longrightarrow (f(u),f(v))\in\theta_2$ and **Mon** the category of monoids. Now a monoid M being given $\theta_M = F(M) = \{(u,v)\in M\,|\,uv=vu\}$ can be checked to be a functor $F:\ \mathbf{Mon}\to\mathbf{CAlph}$

$$\begin{tikzcd}
\mathbf{CAlph} & & \mathbf{Mon} \arrow[ll, dotted, "F"'] \\
(X,\theta) \arrow[rr, dashed, "f"] \arrow[drr, dashed, "j"'] & & M \\
& & M(X,\theta) \arrow[u, "\hat{f}"']
\end{tikzcd}$$

Figure: $M(X,\theta)$ is the monoid freely generated by (X,θ) w.r.t. F. To say that $f\in Het_F((X,\theta),M)$ amounts to say that $f:\ X\to M$ set-theoretically and $(u,v)\in\theta \Longrightarrow f(u)f(v)=f(v)f(u)$

Functor/2: Groups.

㉓ Let **Grp** the category of groups. Now a monoid G being given $\theta_G = F(G) = \{(u, v) \in G \mid uv = vu\}$ can be checked to be a functor $F: \ \mathbf{Grp} \to \mathbf{CAlph}$

$$\begin{tikzcd}
\mathbf{CAlph} & & \mathbf{Grp} \arrow[ll, dotted, "F"'] \\
(X,\theta) \arrow[rr, dashed, "f"] \arrow[drr, dashed, "j"'] & & G \\
& & F(X,\theta) \arrow[u, "\hat{f}"']
\end{tikzcd}$$

Figure: $F(X,\theta)$ is the group freely generated by (X,θ) w.r.t. F. To say that $f \in Het_F((X,\theta), G)$ amounts to say that $f: \ X \to G$ set-theoretically and $(u, v) \in \theta \Longrightarrow f(u)f(v) = f(v)f(u)$.

Functor/3: **k**-Lie algebras.

24. Let **k-Lie** be the category of **k**-Lie algebras (**k** is a ring). Now $L \in$ **k-Lie** being given $\theta_L = F(L) = \{(u,v) \in L \mid [u,v] = 0\}$ can be checked to be a functor $F :$ **k-Lie** $\to$ **CAlph**

$$\begin{tikzcd}
\mathbf{CAlph} & \mathbf{k\text{-}Lie} \arrow[l, dotted, "F"'] \\
(X,\theta) \arrow[r, dashed, "f"] \arrow[dr, dashed, "j"'] & L \\
 & \mathcal{L}ie_k(X,\theta) \arrow[u, "\hat{f}"']
\end{tikzcd}$$

Figure: $\mathcal{L}ie_k(X,\theta)$ is the **k**-Lie algebra freely generated by (X,θ) w.r.t. F.

To say that $f \in Het_F((X,\theta), L)$ amounts to say that $f : X \to L$ set-theoretically and $(u,v) \in \theta \Longrightarrow [f(u), f(v)] = 0$

Functor/4: **k**-AAU.

25 Let **k-AAU** be the category of **k**-algebras (associative with unit) (**k** is a ring). Now $\mathcal{A} \in$ **k-AAU** being given $\theta_{\mathcal{A}} = F(\mathcal{A}) = \{(u,v) \in \mathcal{A} \,|\, [u,v] = 0\}$ can be checked to be a functor $F :$ **k-AAU** $\to$ **CAlph**

$$
\begin{tikzcd}
\mathbf{CAlph} & & \mathbf{k\text{-}AAU} \arrow[ll, dotted, "F"'] \\
(X,\theta) \arrow[rr, dashed, "f"] \arrow[drr, dashed, "j"'] & & M \\
& & \mathbf{k}\langle X,\theta\rangle \arrow[u, "\hat{f}"']
\end{tikzcd}
$$

Figure: $\mathbf{k}\langle X,\theta\rangle$ is the **k**-AAU freely generated by (X,θ) w.r.t. F. To say that $f \in Het_F((X,\theta),\mathcal{A})$ amounts to say that $f : X \to \mathcal{A}$ set-theoretically and $(u,v) \in \theta \Longrightarrow f(u)f(v) = f(v)f(u)$.

Links between these free structures.

26 Let us recall the two functorial paths from **Set** to **k-AAU** constructed in a previous CCRT[n]

$$\begin{array}{lcl} [\mathbf{Set}] \longrightarrow & [\mathbf{Mon}] & \longrightarrow [\mathbf{k-AAU}] \\ [\mathbf{Set}] \longrightarrow & [\mathbf{k-Lie}] & \longrightarrow [\mathbf{k-AAU}] \end{array}$$

27 Here, we observe the same phenomenon

$$\begin{array}{lcl} [\mathbf{CAlph}] \longrightarrow & [\mathbf{Mon}] & \longrightarrow [\mathbf{k\text{-}AAU}] \\ [\mathbf{CAlph}] \longrightarrow & [\mathbf{k\text{-}Lie}] & \longrightarrow [\mathbf{k\text{-}AAU}] \end{array} \tag{14}$$

28 From the first path, we get $\mathbf{k}\langle X,\theta\rangle = \mathbf{k}[M(X,\theta)]$ and from the second $\mathbf{k}\langle X,\theta\rangle = \mathcal{U}(\mathcal{L}ie_{\mathbf{k}}\langle \mathcal{X},\theta\rangle)$.

MRS factorisation.

29. In fact, for all $\mathbf{k}, X$ and θ, the free Lie algebra $\mathcal{L}ie_{\mathbf{k}}\langle X, \theta\rangle$ posesses combinatorial bases. This solves, by Lazard elimination, a conjecture by M.P. Schützenberger [9].
30. The combinatorics of partially commutative Lyndon words was developed at LACIM by P. Lalonde and C. Reutenauer (references on request).
31. In order to perform MRS, we recall its construction for an arbitrary Lie algebra.

MRS: general construction/1

32. Let $\mathbf{k}$ be a $\mathbb{Q}$-AAU and $\mathfrak{g}$ a $\mathbf{k}$-Lie algebra (finite or infinite dimensional), which is free as a k-module. We consider any totally ordered basis $B = (b_i)_{i\in I}$ of $\mathfrak{g}$ ($(I, <)$ with $<$ a strict total ordering). For every $\alpha \in \mathbb{N}^{(I)}$, we set

$$B^\alpha = b_{i_1}^{\alpha_1} b_{i_2}^{\alpha_2} \cdots b_{i_m}^{\alpha_m}$$

with $supp(\alpha) \subset \{i_1 < i_2 < \cdots < i_m\}$ (it is easily checked that B^α is independent from the choice of the "covering" set). For $\alpha \in \mathbb{N}^{(I)}$, let B_α be the linear form of $\mathcal{U}^*(\mathfrak{g})$ defined by $\langle B_\alpha | B^\beta \rangle = \delta_{\alpha,\beta}$. We claim that their convolution (marked with the sign $*$) satisfies

$$B_\alpha * B_\beta = \frac{(\alpha+\beta)!}{\alpha!.\beta!} B_{\alpha+\beta} \tag{15}$$

MRS: general construction/2

33. (Proof) In fact, we have

$$
\begin{aligned}
&\langle B_\alpha * B_\beta | B^\gamma\rangle = \langle B_\alpha \otimes B_\beta | \Delta(B^\gamma)\rangle = \\
&\langle B_\alpha \otimes B_\beta | \sum_{\alpha_1+\beta_1=\gamma} \binom{\gamma}{\alpha_1, \beta_1} B^{\alpha_1} \otimes B^{\beta_1}\rangle = \\
&\sum_{\alpha_1+\beta_1=\gamma} \binom{\gamma}{\alpha_1, \beta_1} \langle B_\alpha \otimes B_\beta | B^{\alpha_1} \otimes B^{\beta_1}\rangle = \\
&\sum_{\alpha_1+\beta_1=\gamma} \binom{\gamma}{\alpha_1, \beta_1} \delta_{\alpha,\alpha_1}\delta_{\beta,\beta_1} = \delta_{\gamma,\alpha+\beta} \binom{\alpha+\beta}{\alpha, \beta}
\end{aligned} \tag{16}
$$

MRS: general construction/3

Proposition

Let $\mathbf{k}$ *be a* $\mathbb{Q}$*-AAU and* $\mathfrak{g}$ *a* $\mathbf{k}$*-Lie algebra which is free as a k-module. Let* $B = (b_i)_{i\in I}$ *an ordered (totally) basis of* $\mathfrak{g}$*. Then*

1. *The space*
$$\mathcal{A} = span_k\{(B_\alpha) | \alpha \in \mathbb{N}^{(I)}\} \subset \mathcal{U}^*(\mathfrak{g}) \tag{17}$$
It is an convolution subalgebra of $(\mathcal{U}^*(\mathfrak{g}), *, \epsilon)$
2. *If* $(\mathcal{B}, \bullet, 1_\mathcal{B})$ *is a commutative algebra every* $\mathcal{B}$*-valued character factorises as the following infinite product*
$$\chi = \prod_{i\in I}^{\rightarrow} e^{\chi(B_{e_i})\, b_i} \tag{18}$$
for the topology of pointwise convergence on $\mathcal{A}$ *(*$\mathcal{B}$ *being discrete) and* e_i *being the elementary basis of* $\mathbb{N}^{(i)}$ *(*$e_i(j) = \delta_{i,j}$*).*

Partially commutative MRS and Möbius function

34. Using Lazard elimination in $\mathcal{L}ie_{\mathbf{k}}\langle X,\theta\rangle$, one can construct all finely homogeneous bases of this Lie algebra, order them arbitrarily (and totally) and apply the preceding construction.
35. Let us now delve in more detail into Cartier and Foata's result about $M(X,\theta)$ Möbius function.
36. Starting with a monoid $(M,\star,1_M)$, considering $\mathbf{k}[M]\subset\mathbf{k}[[M]]=\mathbf{k}^M$, we see that in order to extend the product formula

$$P\star Q:=\sum_{uv=w}\langle P|u\rangle\langle Q|v\rangle\, w \tag{19}$$

it is sufficient (and necesary in general position) that the map $\star : M\times M\to M$ has finite fibers[a]

[a] Recall that a map $f:\ X\to Y$ between two sets X and Y has finite fibers if and only if for each $y\in Y$, the preimage $f^{-1}(y)$ is finite.

37. In this case then we can extend the formula (19) to arbitrary $P, Q \in \mathbf{k}^M$ (as opposed to merely $P, Q \in \mathbf{k}[M]$). In this case, the $\mathbf{k}$-algebra $\left(\mathbf{k}^M, \star, 1_M\right)$ is called the total algebra of M, [a] and its product is the Cauchy product between series.
38. For every $S \in \mathbf{k}^M$, the family $(\langle S|m\rangle\, m)_{m\in M}$ is summable[b]. and its sum is $S = \sum_{m\in M} \langle S|m\rangle\, m$.

[a] See also `https://en.wikipedia.org/wiki/Total_algebra`.

[b] We say that a family $(a_s)_{s\in S}$ of elements of $\mathbf{k}^M$ is summable if for any given $n \in M$, all but finitely many $s \in S$ satisfy $\langle a_s|n\rangle = 0$. Such a summable family will always have a well-defined infinite sum $\sum_{s\in S} a_s \in \mathbf{k}^M$, whence the name “summable”.

Partially commutative Möbius function/2

39 For every series $S \in \mathbf{k}[[M]]$, we set $S_+ := \sum_{m \neq 1} \langle S|m \rangle\, m$.
In order for the family $((S_+)^n)_{n \geq 0}$ to be summable, it is sufficient that the iterated multiplication map $\mu^* : \ (M_+)^* \to M$ defined by

$$\mu^*[m_1, \dots, m_n] = m_1 \cdots m_n \text{ (product within } M) \tag{20}$$

have finite fibers (where we have written the word $[m_1, \dots, m_n] \in (M_+)^*$ as a list to avoid confusion).[a]

40 In this case the characteristic series of M (i.e. $\underline{M} = \sum_{m \in M} m = 1 + \underline{M_+}$) is invertible and

$$\underline{M}^{-1} = 1 - \underline{M_+} + \underline{M_+}^2 - \underline{M_+}^3 - \cdots = \sum_{m \in M} \mu(m).m \tag{21}$$

[a]Furthermore, this condition is also necessary (if S_+ is generic) if $\mathbf{k} = \mathbb{Z}$. These monoids are called "locally finite" in [15].

Partially commutative Möbius function/3

41. $\mu : \; M \to \mathbb{Z}$ is called the Möbius function of M.
42. The Möbius function of the multiplicative monoid $(\mathbb{N}_{\geq 1}, \times)$ is well-known. If

$$n = \prod_{p \in \mathfrak{P}} p^{\nu_p(n)}$$

$\mu(n) = 0$ if one of the factors $\nu(p) \geq 2$ (n contains a square) and

$$\mu(n) = (-1)^{|supp(p \mapsto \nu_p(n))|}$$

otherwise.

43. It is a particular case of Cartier-Foata theorem [5].

Partially commutative Möbius function/4

44. The result is

$$\sum_{m\in M} \mu(m).m = \underline{M(X,\theta)}^{-1} = \sum_{C \text{ clique of } \theta} (-1)^{|C|}\underline{C} \qquad (22)$$

where $\underline{C}$ is the product of the elements of $C \subset X$.

45. **Examples.** –
i) $\theta = \Delta_X = \{(x,x)\}_{x\in X}$, then $M(X,\theta) = X^*$, we have

$$(X^*)^{-1} = 1 - X \qquad (23)$$

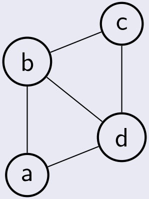
ii) $\theta = X \times X$ then $MX,\theta)$ is the free commutative monoid and

$$\underline{M(X,\theta)}^{-1} = \prod_{x\in X} (1-x) \qquad (24)$$

Partially commutative Möbius function/4

44. The Möbius function is non-zero only for the square-free words w and, in this case, its value is $\mu(w) = (-1)^{|w|}$.

Partially commutative Möbius function/5



45 For this graph, we have

$$\underline{M(X,\theta)}^{-1} = 1 - a - b - c - d + ab + ad + bc + bd + cd - abd - bcd \quad (25)$$

46 and, then each commutative character of $\mathbf{k}\langle X,\theta\rangle$ is a star of the form

$$\chi = \big(\chi(a).a + \chi(b).b + \chi(c).c + \chi(d).d - \chi(a)\chi(b)ab$$
$$-\chi(a)\chi(d)ad - \chi(b)\chi(c)bc - \chi(b)\chi(d)bd - \chi(c)\chi(d)cd$$
$$+\chi(a)\chi(b)\chi(d)abd + \chi(b)\chi(c)\chi(d)bcd\big)^*$$

Concluding remarks

1. We have seen Free Partially Commutative Structures over the categories **Mon**, **Grp**, **k-Lie**, **k-AAU**.
2. These structures are reviewed within several mathematical papers the most (pre-)categorical one being [10] (other references available on request).
3. They share many features with the free ones and interpolate between commutative and noncommutative worlds.
4. To cite only a few: Magnus theory (Magnus transformation $x \mapsto 1 + x$), Lower central series of the Free Group, Free Lie algebra within the polynomials, Lyndon words and bases, Lazard elimination, Lazard codes and Hall bases, Free decompositions of the Monoid, Group, Lie algebra and associative algebra.

Concluding remarks

5. A closure theorem exists saying that (in characteristic zero), it is the maximal framework where Magnus theory holds.
6. Next time, we will speak about universal constructions on differential modules, localization and wronskians.

Thank you for your attention.

Links

1. Categorical framework(s)

   `https://ncatlab.org/nlab/show/category`
   `https://en.wikipedia.org/wiki/Category_(mathematics)`

2. Universal problems

   `https://ncatlab.org/nlab/show/universal+construction`
   `https://en.wikipedia.org/wiki/Universal_property`

3. Paolo Perrone, *Notes on Category Theory with examples from basic mathematics*, 181p (2020)
   arXiv:1912.10642 [math.CT]

   `https://en.wikipedia.org/wiki/Abstract_nonsense`

4. Heteromorphism

   `https://ncatlab.org/nlab/show/heteromorphism`

5. D. Ellerman, *MacLane, Bourbaki, and Adjoints: A Heteromorphic Retrospective*, David EllermanPhilosophy Department, University of California at Riverside

Links/2

6. https://en.wikipedia.org/wiki/Category_of_modules
7. https://ncatlab.org/nlab/show/Grothendieck+group
8. Traces and hilbertian operators

   https://hal.archives-ouvertes.fr/hal-01015295/document
9. State on a star-algebra

   https://ncatlab.org/nlab/show/state+on+a+star-algebra
10. Hilbert module

    https://ncatlab.org/nlab/show/Hilbert+module

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[30] Definition of (commutative) free augmented algebras
https://mathoverflow.net/questions/352726

[31] Closed subgroup (Cartan) theorem without transversality nor Lipschitz condition within Banach algebras
https://mathoverflow.net/questions/356531

[32] Definition of augmented algebras (general)
https://ncatlab.org/nlab/show/augmented+algebra

[33] Kernels in nlab
https://ncatlab.org/nlab/show/kernel